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ABSTRACT

The presence of illiquid assets, such as human wealth, housing and a proprietorship substantially complicates the problem of portfolio choice. This paper is concerned with the problem of optimal asset allocation and consumption in a continuous time model when one asset cannot be traded. This illiquid asset, which depends on an uninsurable source of risk, provides a liquid dividend. In the case of human capital we can think about this dividend as labor income. The agent is endowed with a given amount of the illiquid asset and with some liquid wealth which can be allocated in a market where there is a risky and a riskless asset. The main point of the paper is that the optimal allocations to the two liquid assets and consumption will critically depend on the endowment and characteristics of the illiquid asset, in addition to the preferences and to the liquid holdings held by the agent. We provide what we believe to be the first analytical solution to this problem when the agent has power utility of consumption and terminal wealth. We also derive the value that the agent assigns to the illiquid asset. The risk adjusted valuation procedure we develop can be used to value both liquid and illiquid assets, as well as contingent claims on those assets.

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1 Introduction

The problem of optimal asset allocation is of great importance both in the theory as well as in the practice of finance. Since the seminal work of Markowitz (1952) scholars and practitioners have looked at the issue of how much money should an investor optimally allocate to different assets or asset classes. The single period model of Markowitz (1952) was extended to a multiperiod setting by Samuelson (1969) and then to continuous time by Merton (1969, 1971). The traditional approach assumes that all assets can be traded at all times. This paper is concerned with optimal asset allocation in a continuous time model when one asset cannot be traded.

Typical examples of assets in which trading is problematic include human wealth, housing and a proprietorship. When the asset allocation problem is solved without taking into account the existence of these "illiquid"assets the allocation is certainly suboptimal. Consider the following example.[a] Two individuals with the same wealth, the same preferences and the same horizon would invest in the same portfolio using the traditional asset allocation framework. However, if one of the individuals is a stock broker with his human wealth highly correlated with the stock market, and the other is a tenured university professor with his human wealth independent of the stock market, it would be reasonable to expect that they would have different allocations. This is the problem we address in this paper.

There are many definitions of illiquid assets. To make the problem tractable, in this paper we assume that illiquidity prevents the trading of the asset over the time horizon we consider (though this time horizon could become infinite). The illiquid asset, however, provides a liquid "dividend" that is related to the level of an observable state variable associated with the illiquid asset. In the case of human wealth the dividend could be labor income, in the case of housing the dividend could be the housing services, and in the case of a proprietorship it could be distributed profits from the business. The uncertainty that drives the illiquid asset cannot be fully diversified in the market. Since the asset is not traded, the state variable associated with the illiquid asset can not be interpreted as a price. In the finite horizon case this state variable becomes the price of the asset only at the terminal date.

We assume that the agent in endowed with a given amount of the illiquid asset and with some liquid wealth which can be allocated in a market where there are two liquid assets: a risky asset and a risk free asset. The main point of this paper is that the allocations to the two liquid assets and consumption

[a]This example was presented by Robert Merton in a talk in Verona, Italy in June 2003.

will critically depend on the endowment and characteristics of the illiquid asset, in addition to the preferences and liquid wealth of the agent.

In the process of establishing the optimal allocation to the liquid assets we also derive the value that the agent assigns to the illiquid asset. As expected, the value that the agent assigns to the illiquid asset will always be lower than the value it would have if it were traded. Moreover, this value depends not only on the level of the state variable associated with the illiquid asset, but also on the preferences of the agent.

The problem of asset allocation in the presence of illiquid assets has been the subject of intense research in the finance literature since the late 60's. Recognizing the complexity of the subject simplifying assumptions have been introduced to make the problem tractable.

Among non tradable assets, human wealth is certainly the most relevant source of risk in the individual allocation problem which is difficult to insure or diversify. Bodie, Merton and Samuleson (1992) consider a long horizon investor with a riskless stream of labor income and show that an investor with riskless non tradable human wealth should tilt his financial portfolio toward stocks relative to an investor who owns only tradable stock. Jagannathan and Kocherlakota (1996) show that this advice is economically sound as long as the human wealth is relatively uncorrelated with stock returns. Zeldes (1989) performs a numerical study of a discrete time model of optimal consumption in the presence of stochastic income. Koo (1995) and Heaton and Lucas (1997) introduce risky labor income and portfolio constraints in an infinite horizon portfolio choice problem and, using a numerical simulation, focus on how the presence of background risks from sources such as labor, influences consumption and portfolio choice. Both papers find that investors hold most of their financial wealth in stocks. Koo (1995) shows numerically that an increase in the variance of permanent income shocks decreases both the optimal portfolio allocation to stocks and the consumption labor income ratio of power utility investors. In a discrete time framework Viceira (2001) considers an approximate solution and finds that positive correlation between labor income innovations and unexpected financial returns reduces the investor's willingness to hold liquid risky asset because of its poor properties as an hedge against unexpected declines in labor income. Consistently, Heaton and Lucas (2000) find that entrepreneurs have significantly safer portfolios of financial assets than other investors with similar wage and wealth. Campbell and Viceira (2002) provide a comprehensive discussion of the empirical testing and of the economic implications of including human wealth in the household portfolio choice problem. Dybvig and Liu (2004) consider a lifetime consumption/investment model with endogenous retirement date.

Wang (2006) provides a complete discussion of the consumption choices of an agent with human and liquid wealth under the assumption of exponential utility function.

A related literature deals with portfolio choice in the presence of assets which cannot be traded. To our knowledge the first treatment is due to Myers (1972, 1973) which solves the static version of the problem. In a dynamic context, the problem we solve can be seen as the limit of large transaction costs of the Grossmann and Laroque (1990) model for illiquid durables. Svensson and Werner (1993) provide a treatment with exponential preferences. Longstaff, Liu and Kahl (2003) formulate and provide a numerical solution to the optimal dynamic allocation problem of an investor with power utility whose portfolio includes a stock which cannot be sold. Among the possible sources of background risk in household portfolios, housing is certainly one important asset class that is relatively illiquid and undiversified. Analyzing risk and return is however complicated because of the unobservable flow of consumption of housing services. Flavin and Yamashita (1998) consider housing both as an asset and as a source of consumption, and obtain the optimal portfolio allocations by simulation.

There is a large strand of the literature in stochastic optimization which addresses the continuous time portfolio allocation problem in incomplete markets both with the direct partial differential equation approach and with the martingale-measure duality approach. Duffie, Fleming, Soner and Zariphopolou (1997) study an asset allocation problem for an investor which maximizes HARA utility (with relative risk aversion coefficient smaller than 1) from consumption in a market composed by a risky and a riskless asset and receives an income which cannot be replicated by other securities. This study proves existence, uniqueness and regularity of the value function, while the optimal consumption path and the allocation strategy are implicitly specified throughout a feedback expression. Koo (1998) analyzes the same problem in the presence of constraints.

The stochastic optimization problem we discuss is strictly related to the utility based pricing of contingent claims whose underlying assets are non traded. Most of these references, Davis (1999), Detemple and Sundaresan (1999), Teplà (2000), Hobson and Henderson (2002), Henderson (2002) and Musiela and Zariphopolou (2004a)), assume that the agent has exponential preferences and no consumption and dividends.

Our results are based on the duality approach pioneered by Cox and Huang (1989), He and Person (1991), and Karatzas et al. (1991). He and Pages (1993) and El Karoui and Jeanblanc (1998) deal with a constrained version of the problem when labor income risk can be diversified in the market.

When the agent receives an uninsurable random endowment the mathematical formulation of the stochastic control problem becomes difficult. Existence results under very general conditions on the price processes and on the utility function have been obtained by Cuoco (1997) attacking directly the primal problem, while exact results on the duality approach have been established by Cvitanic, Schachermayer and Wang (2001) in the case of maximization of utility from terminal wealth and extended by Karatzas and Zitkovic (2003) to the problem with intertemporal consumption and constraints.

As far as we are aware, our paper contains the first analytical solution to this problem when the agent has power utility of consumption and terminal wealth. The analytical solution obtained allows us to quantify the impact of the assets characteristics and the agent preferences on optimal asset allocation and consumption. In particular, we show that the higher is the correlation between the liquid and the illiquid asset, the lower will be the allocation to the risky liquid asset. So, in the example given above the professor would optimally invest a higher proportion of his liquid wealth in the risky liquid asset than the stock broker. Since the human wealth of the stock broker is highly correlated with the stock market, and his human wealth is non tradable, he will tend to invest a smaller fraction of his liquid wealth in the market portfolio.

The analysis of the optimal consumption out of liquid assets shows that the agent will make his consumption decision looking not only at his liquid wealth but taking into account also his illiquid wealth. The computation of the elasticity of consumption with respect to liquid and illiquid wealth shows that for an agent with constant relative risk aversion the propensity to consume out of liquid wealth will be always larger than out of illiquid wealth.

Moreover, the solution provides the expression for the stochastic discount factor that the agent uses for his/her private valuation of the illiquid asset. The procedure reduces to risk neutral valuation for the liquid assets, while the risk adjustment for valuing illiquid assets is found to be state dependent. The optimal value function of the allocation/consumption problem determines the analytical expression of the risk adjustment as a function of the liquidity state variable, the preferences of the investor and the volatility of the non-traded source of risk.

The paper is organized as follows. In Section 2 we introduce the economic setting and its mathematical formulation, in Section 3 we provide the analytical solution to the Hamilton Jacobi Bellman equation. Section 4 provides a discussion of the optimal consumption and allocation policies. Section 5 derives the stochastic discount factor that the agent will use in the private valuation of the illiquid asset. In Section 6 we analyze the sensitivity of the control policies to different parameters and Section 7 concludes. The Appendix contains the proofs and some technical results.

2 Economic setting

Consider a continuous time economy where prices evolve stochastically in a filtered probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$ supporting a two dimensional Brownian motion (W_t^1, W_t^2) where $\mathcal{F} = \{\mathcal{F}_t\}_{t \leq T}$ and $\mathcal{F}_t$ represents the augmented filtration generated by all the information reflected in the market up to time t and $\mathbb{P}$ is the objective probability measure. All the processes will be assumed to be adapted to $\mathcal{F}$. We fix a final time horizon T, the epoch at which the non traded (illiquid) asset becomes tradable and can be consumed.

The market is composed of three assets:

- The risk free bond B_t, whose dynamics is:

$$dB_t = rB_t dt \quad t \leq T$$

  where r is the continuously compounded risk free interest rate which, for simplicity, we assume to be constant.

- A traded liquid risky asset S_t, whose dynamics is:

$$\frac{dS_t}{S_t} = \alpha dt + \sigma dW_t^1 \quad t \leq T$$

  where α $(> r)$is the continuously compounded expected rate of return on the risky liquid asset, and σ is the continuous standard deviation of the rate of return.[b]

- An illiquid risky asset H_t (no trading is allowed until time T, when it can be consumed), whose dynamics is:

$$\frac{dH_t}{H_t} = (\mu - \delta)dt + \eta\left(\rho dW^1 + \sqrt{1-\rho^2}dW^2\right) \quad t \leq T \qquad (1)$$

  where μ is the continuously compounded total expected rate of return on the risky illiquid asset, δ is the liquid continuous rate of dividend paid by the illiquid asset, η is the continuous standard deviation of the

[b]For simplicity we assume that the liquid risky asset pays no dividends, but the analysis would be the same if the asset paid a continuous dividend

rate of return, and ρ is the correlation coefficient between the dynamics of the liquid and the illiquid risky asset.[c]

Since the illiquid asset H_t cannot be traded at any time $t < T$, it represents the level of a state variable associated with the illiquid asset, i.e. the process which generates the random cashflows: δH_t at time $t < T$ and H_T at time T. At time T, the state variable, H_T, becomes equal to the price of the illiquid asset.

We assume that the investor has time additive separable preferences and maximizes a CRRA utility from consumption and final wealth. The intertemporal optimization problem of the agent is then given by:

$$\begin{aligned} & \sup_{(\pi,c)\in A(l,h,t)} \mathcal{U}\left(t,c,W_T^{\pi,c}\right) \\ \mathcal{U}\left(t,c,W_T^{\pi,c}\right) \quad &: \quad = E_t^{\mathbb{P}}\left[\int_t^T e^{-\kappa(u-t)} U_\gamma\left(c_u\right) du + \beta e^{-\kappa(T-t)} U_\gamma\left(W_T^{\pi,c}\right)\right] \\ with \quad U_\gamma\left(x\right) \quad &= \quad \frac{x^{1-\gamma}}{1-\gamma} \quad for\ x>0, \quad U_\gamma\left(x\right) = -\infty \quad for\ x \leq 0, \end{aligned}$$

The set of admissible plans with initial liquid wealth l, and initial level of the illiquid state variable h, $A\left(l,h,t\right)$, is defined as the set of admissible consumption-allocation plans (c,π). [1] The consumption stream, $c \equiv \left(c_\tau\right)_{t\leq\tau\leq T}$, specifies the agent's consumption rate of liquid assets, while the allocation strategy $\pi = \left(\pi_\tau^S, \pi_\tau^B\right)_{t\leq\tau\leq T}$ denotes the dollar amounts invested in the risky and riskless liquid assets at any time τ between t and T. Assume that at time t the agent holds an amount of liquid wealth $l>0$, and a nominal amount of illiquid asset $H_t = h \geq 0$, then (c,π) is admissible if there exists a strategy π which finances a consumption stream c i.e.

$$\begin{aligned} L_t &= l, \\ L_\tau &= \pi_\tau^S + \pi_\tau^B, \quad t\leq\tau\leq T \end{aligned} \tag{2}$$

and the dynamics of liquid wealth is: [2]

$$dL_\tau = \left(rL_\tau + \delta H_\tau - c_\tau\right) d\tau + \pi_\tau^S \left(\frac{dS_\tau}{S_\tau} - r d\tau\right).$$

At time T the investor will consume his total wealth, $W_T^{\pi,c} = L_T + H_T$ which includes the liquid wealth, L_T, and the illiquid wealth, H_T.

[c]Since human wealth will enter in the state equation for liquid wealth equation only through dividends, the dividend plays exactly the same role of a stochastic income for the agent. For this reason, our allocation problem can be considered as the finite horizon counterpart of an allocation problem in the presence of an uninsurable stochastic income.

3 Solution of the stochastic optimization problem

We search for the value function defined by:

$$V(l,h,t) := \sup_{(\pi,c)\in A(l,h,t)} \mathcal{U}\left(t,c,W_T^{\pi,c}\right) \tag{3}$$

The Hamilton Jacobi Bellman (HJB) equation corresponding to the above stochastic optimization problem can be written as:

$$\begin{aligned} 0 &= V_t(l,h,t) + \frac{1}{2}\sigma^2 h^2 V_{hh}(l,h,t) + (rl+\delta h)V_l(l,h,t) \\ &\quad + (\mu-\delta)hV_h(l,h,t) + \max_{\pi^S} G\left[\pi^S\right] + \max_{c\geq 0} H[c] \\ G[\pi] &= \frac{1}{2}\left(\pi^S\right)^2\sigma^2 V_{ll}(l,h,t) + \pi^S\eta\rho\sigma h V_{lh}(l,h,t) + (\alpha-r)\pi^S V_l(l,h,t) \\ H[c] &= -cV_l(l,h,t) + \frac{c^{1-\gamma}}{1-\gamma} \end{aligned} \tag{4}$$

when $l,h>0$. With boundary condition:

$$V(l,h,T) = \beta U_\gamma(L_T+H_T)$$

The optimal allocation and consumption strategies will be obtained in feedback form solving:

$$\begin{aligned} c_*(l,h,t) &= \arg\max_{c\geq 0} H(c) \\ &= V_l(l,h,t)^{-1/\gamma} \\ \pi_*^S(l,h,t) &= \arg\max_{\pi^S} G\left(\pi^S\right) \\ &= -\frac{\alpha-r}{\sigma}\frac{V_l(l,h,t)}{V_{ll}(l,h,t)} - \frac{\eta\rho}{\sigma}h\frac{V_{lh}(l,h,t)}{V_{ll}(l,h,t)} \end{aligned} \tag{5}$$

3.1 Homogeneity transformation

In order to reduce the number of state variables in the HJB equation from two to one we conjecture that the value function that solves eq. (4) has the form:

$$\begin{aligned} V(l,h,t) &= e^{-\kappa t}h^{1-\gamma}V(z,t) \\ z &= l/h \end{aligned} \tag{6}$$

and verify that we obtain a single variable HJB equation for the reduced value function $V(z,t)$. Inserting the expression (6) in (4) and performing the maximizations in (5) we obtain that $V(z,t)$, $z>0$, has to obey:

$$
\begin{aligned}
0 \;=\;& V_t(z,t) + K_1 V(z,t) + K_2 z V_z(z,t) - K_3 \frac{(V_z(z,t))^2}{V_{zz}(z,t)} + \\
& + K_4 z^2 V_{zz}(z,t) + F(V_z(z,t))
\end{aligned}
\tag{7}
$$

where:

$$F(x) = -\frac{x^b}{b} + \delta x \qquad b = 1 - \frac{1}{\gamma}$$

with boundary condition:

$$V(z,T) = \beta U_\gamma (1+z_T), \quad z_T = L_T/H_T$$

and where the coefficients are given by:

$$
\begin{aligned}
K_1 &= -\kappa + (\mu-\delta)(1-\gamma) - \frac{1}{2}(1-\gamma)\gamma\eta^2 \\
K_2 &= -(\mu-\delta) + r + (\alpha - r)\frac{\eta\rho}{\sigma} + \gamma\eta^2\left(1-\rho^2\right) \\
K_3 &= \frac{1}{2}\left(\frac{(\alpha-r)}{\sigma} - \eta\rho\gamma\right)^2 \\
K_4 &= \frac{1}{2}\eta^2\left(1-\rho^2\right)
\end{aligned}
$$

Since this equation is highly non linear we use a duality transformation to solve it.

3.2 Duality Transformation

In order to solve the above HJB equation for $V(z,t)$ we need consider a change of variables: we introduce the dual value function for the consumption investment problem, $\widetilde{V}(y,t)$, which is related to $V(z,t)$ by the transform: [3]

$$\widetilde{V}(y,t) := \max_{z\in\mathbb{R}^+} \{V(z,t) - zy\} \tag{8}$$

At optimality the first order condition, $y = V_z(z,t)$, provides a one to one relation between the level of the dual variable y and the level of the state variable in the direct problem z. The new variable y can be interpreted as a marginal utility level rescaled by a factor $h^{-\gamma}$.

Now we show that the HJB for $V(z,t)$ implies an HJB also for the dual value function $\widetilde{V}(y,t)$. Differentiating the above convex duality relation (8), at optimality, the following relations between the derivatives of $V(z,t)$ and $\widetilde{V}(y,t)$ hold:

$$y = V_z(z,t), \quad \widetilde{V}_y(y,t) = -z, \quad V_{zz}(z,t) = -\frac{1}{\widetilde{V}_{yy}(y,t)} \tag{9}$$

where the arguments y and z are related by the optimality condition $y = V_z(z,t)$ and the last equation is obtained from the first two relations as follows:

$$V_{zz}(z,t) = \frac{dV_z(z,t)}{dz} = -\frac{dy}{d\widetilde{V}_y(y,t)} = -\frac{1}{\widetilde{V}_{yy}(y,t)}$$

Direct substitution of these relations in the HJB equation (7) for $V(z,t)$ implies that $\widetilde{V}(y,t)$ is determined by the (convex dual) HJB differential equation:

$$\begin{aligned} 0 \;=\;& \widetilde{V}_t(y,t) + K_1\widetilde{V}(y,t) - (K_1+K_2)\,y\widetilde{V}_y(y,t) + K_3 y^2 \widetilde{V}_{yy}(y,t) + \\ & -K_4\frac{\left(\widetilde{V}_y(y,t)\right)^2}{\widetilde{V}_{yy}(y,t)} - \frac{y^b}{b} + \delta y \end{aligned} \tag{10}$$

with boundary conditions:

$$\widetilde{V}(y,T) = -\beta^{1/\gamma}\frac{y^b}{b} + y$$

Notice that given the solution for $\widetilde{V}(y,t)$, the direct value function $V(z,t)$ is recovered through the inverse transform:

$$V(z,t) = \min_{y\in\mathbb{R}^+}\left\{\widetilde{V}(y,t) + zy\right\} \tag{11}$$

and the level of the state variable z is related to that of the dual state variable y by:

$$z(y,t) := -\widetilde{V}_y(y,t)$$

3.3 Series expansion solution of the dual HJB

The HJB (10) is a quasi linear equation and remarkably we will be able to characterize its solution in terms of a series expansion. We guess a solution of the form: [4]

$$\widetilde{V}(y,t) = y^{1-\frac{1}{\gamma}}\left\{B_0(t) + \sum_{n=1}^{\infty} y^{\frac{n}{\gamma}} B_n(t)\right\} \tag{12}$$

It is easy to show that the first term corresponds exactly to the Merton solution in complete markets.

The solution is thus completely specified by the computation of the coefficients $\{B_n(t)\}_{n\in\mathbb{N}}$. The boundary conditions will be:

$$\begin{aligned} B_0(T) &= -\frac{\beta^{1/\gamma}}{b}, \\ B_1(T) &= +1, \\ B_n(T) &= 0 \qquad n \geq 2, \end{aligned}$$

The explicit computation of the coefficients can be performed inserting the formal expansion (12) in the HJB equation (10) and requiring that the equivalent equation (for $\widetilde{V}_{yy}(y,t) \neq 0$):

$$\begin{aligned} 0 &= \left[\widetilde{V}_t(y,t) + K_1\widetilde{V}(y,t) - (K_2+K_1)\,y\widetilde{V}_y(y,t) + K_3y^2\widetilde{V}_{yy}(y,t)\right] y^2\widetilde{V}_{yy}(y,t) \\ &+ F(y)\,y^2\widetilde{V}_{yy}(y,t) - K_4y^2\left(\widetilde{V}_y(y,t)\right)^2 \end{aligned}$$

is verified at any order.

In the Appendix we show the result that after the insertion of the expression (12) the HJB equation becomes equivalent to a system of linear ODE with respect to time which uniquely determine the coefficients $\{B_n(t)\}_{n\in\mathbb{N}}$.

This system of ODE can be solved iteratively and provides the expression for the dual value function. Then we can state the following:

Proposition 1 *The HJB (10) is solved by:*

$$\widetilde{V}(y,t) = y^{1-\frac{1}{\gamma}}\left\{B_0(t) + \sum_{n=1}^{\infty} y^{\frac{n}{\gamma}} B_n(t)\right\}$$

with:

$$\begin{aligned} B_0(t) &= \left\{\exp\left[a_0(T-t)\right]\left(\beta^{1/\gamma} + \frac{1}{a_0}\right) - \frac{1}{a_0}\right\}\left(-\frac{1}{b}\right) \\ B_1(t) &= \exp\left[a_1(T-t)\right]\left(1 + \frac{\delta}{a_1}\right) - \frac{\delta}{a_1} \\ B_n(t) &= -K_4\int_t^T e^{a_n(s-t)} f^{(n)}(s)\,ds \quad n \geq 2 \end{aligned}$$

where $f^{(n)}(s)$ is given in eq.(23) of the Appendix and can be determined recursively in terms of the functions $\{B_k(t)\}_{k=1,..,n-1}$. The coefficients a_i are *given by:* [5]

$$
\begin{aligned}
a_0 &= K_1 - (K_2 + K_1)\, b + K_3 \left(b^2 - b\right) - K_4 b / (b-1) \\
a_1 &= 2K_4 / (b-1) - K_2 \\
a_n &= K_1 - (K_2 + K_1 - 2K_4/(b-1))\, C_n + \left(K_3 + K_4/(b-1)^2\right) D_n \quad n \geq 2 \\
C_n &= b + n/\gamma, \quad D_n = C_n (C_n - 1), \quad b = 1 - \frac{1}{\gamma}
\end{aligned}
$$

Proof. The proof of the proposition is given in the Appendix. ■

Now that we have succeeded in finding a solution to the dual problem we can determine the solution for the primal through (11) and finally using relation (6) we obtain the value solution to the original problem we addressed (4). [6]

4 Analysis of optimal consumption and allocation policies

Having solved for the value function it is easy to proceed to determine the optimal consumption and asset allocation using equations (5). To provide for a better understanding of the solution obtained and to be able to relate our results to those existing in the current literature, we introduce a representation that allows us to get a better intuition of how the agent 'values' his illiquid wealth and how his decisions are related to it.

4.1 Marginal valuation of the illiquid asset

The marginal utility based value of the illiquid asset p is defined as the liquid amount at which the investor would be willing to sell an infinitesimal amount ε of the illiquid asset at time t and is defined by:

$$
\begin{aligned}
V(l, h+\varepsilon, t) &\approx V(l + p\varepsilon, h, t) + O\left(\varepsilon^2\right) \\
V_h(l, h, t) &= p V_l(l, h, t) \\
p &:= \frac{V_h(l, h, t)}{V_l(l, h, t)}
\end{aligned}
$$

The homogeneous representation of the value function, eq.(6), jointly with eqs.(24) in the Appendix show that p is a function of the state variable $z = l/h$ and its expression in terms of the reduced value function $V(z, t)$ is given by:

$$
p(z, t) = (1-\gamma) \frac{V(z, t)}{V_z(z, t)} - z
$$

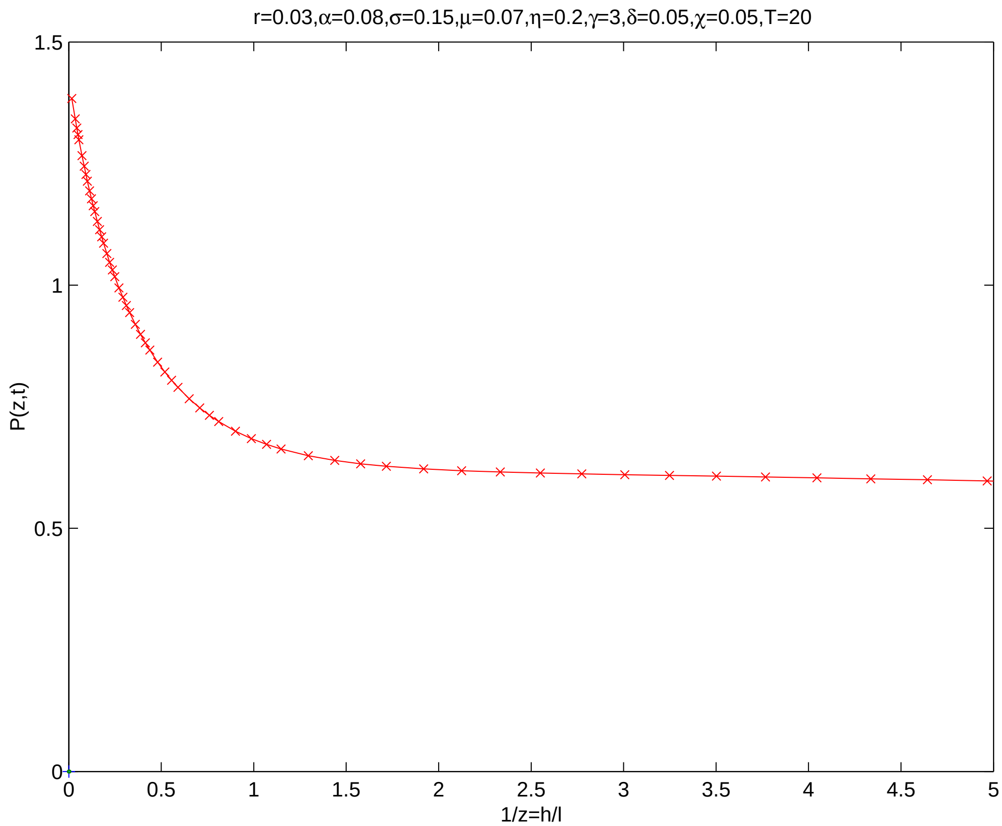

Figure 1: Marginal utility based price as a function of the ratio $1/z = h/l$.

In the case that the illiquid asset is private equity, the marginal utility based value of the illiquid asset $p(z,t)$ corresponds to the minimum price the agent would be willing to receive in order to sell an infinitesimal amount of the private equity.

For the purpose of comparing our analysis with the existing literature, in particular with Merton (1971) it is useful to define:

$$w(l,h,t) = l + p(z,t)\,h = h(z + p(z,t)) \tag{13}$$

$w(l,h,t)$ would be the equivalent total liquid wealth if the agent would value his illiquid holding at the marginal value $p(z,t)$. Following Koo (1998) we refer to $w(l,h,t)$ as *implicit total wealth*. Note that since $p(z,t)$ is a function of z the total value that the investors assigns to the illiquid asset will be different from $p(z,t)\,h$. In Section 5 we return to this issue.

To gain insight into the results obtained we illustrate our findings through a numerical example. Consider an investor with an horizon of 20 years, a coefficient of risk aversion of $\gamma = 3$, and $\beta = 1$ who is allocating funds to a liquid risky asset with an expected rate of return of $\alpha = 0.08$ and volatility $\sigma = 0.15$. In addition he holds an illiquid risky asset with a drift $\mu = 0.07$ and volatility $\eta = 0.20$. Even though the illiquid asset can not be traded, it pays a liquid dividend yield of $\delta = 0.05$. Finally, there is a liquid riskless asset with a constant interest rate $r = 0.03$. In the illustration we consider as liquidity state variable h/l, the proportion of illiquid wealth per unit of liquid wealth within the range of parameters where the nominal proportion of illiquid wealth varies from 0 to 5 times the liquid holdings of the agent. In Table I we report the parameters we used in the illustration.

Table I

T	α	σ	μ	η	ρ	δ	r	κ	γ	β
20	0.08	0.15	0.07	0.2	0.2	0.05	0.03	0.05	3	1

Figure 1 shows the marginal utility based value as a function of liquidity expressed as the proportion of illiquid wealth per unit of liquid wealth, $1/z = h/l$, for the parameters given in Table I. When $h \to 0$, p converges to a constant implying that the investor would be willing to sell at this price his total illiquid wealth, which at this point becomes an infinitesimal fraction of liquid one (in Section 5 we compute this price). Increasing h/l, p decreases monotonically, hence the more illiquid the agent, the less valuable is an infinitesimal share of his illiquid wealth. When $h/l \to \infty$, p also converges to a constant, which is the price the investor would be willing to sell an infinitesimal amount of his illiquid wealth when his liquid holdings are vanishing.

4.2 Optimal consumption over wealth ratio

For purposes of comparison it is useful to define the consumption over implicit total wealth ratio as:

$$q := \frac{c^*(l,h,t)}{w(l,h,t)}$$

where $c^*(l,h,t) = V_l(l,h,t)^{-1/\gamma}$ is the expression for optimal consumption obtained maximizing eqs.(5). An easy verification shows that also q is independent of h and is a function of z and t only.

In the appendix we prove that the value function can be expressed in terms of $q(z,t)$ and $w(l,h,t)$ as:

$$V(l,h,t) = q^{-\gamma}(z,t)\,\frac{w(l,h,t)^{1-\gamma}}{1-\gamma} \tag{14}$$

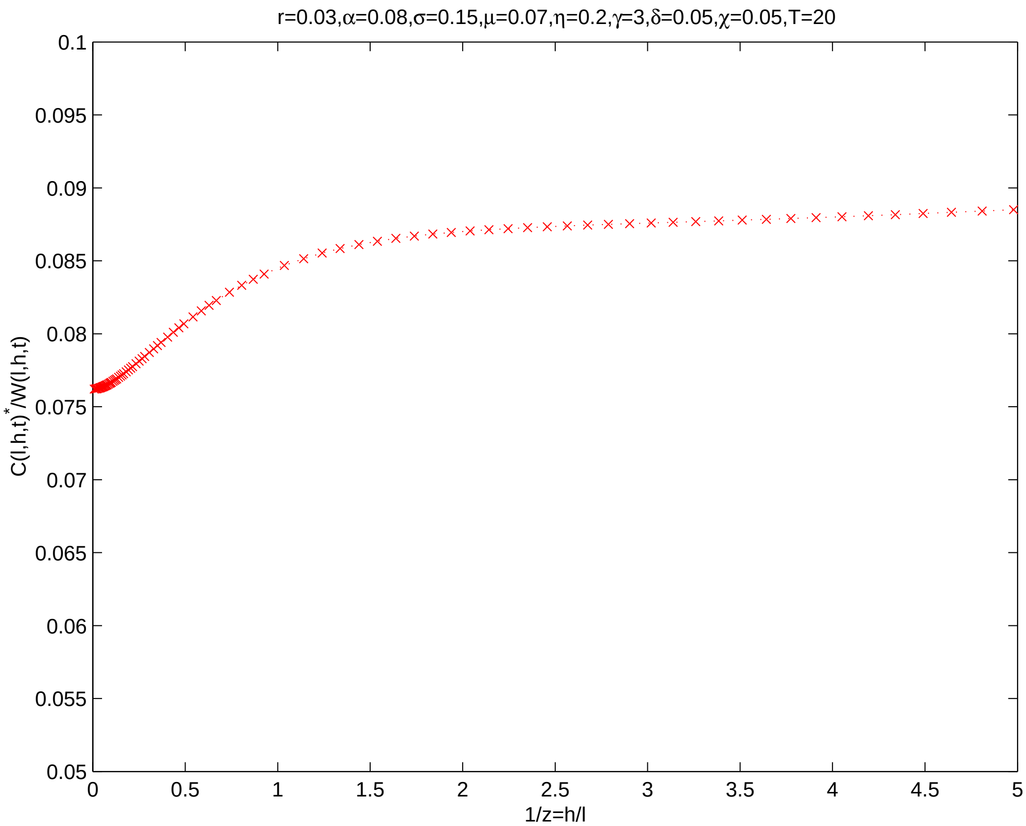

Figure 2: Consumption over implicit total wealth $1/z = h/l$

This representation shows that for a fixed value of the state variable z the value function assumes the standard Merton form, where $w(l,h,t)$ is now the implicit valuation of the agent total wealth. Similarly, a liquidity constrained investor will consume a fraction $q(z,t)$ of his implicit total wealth. In Figure 2 we show the value of q as a function of liquidity $1/z = h/l$ for the parameters given in Table I. The limit value of $q(z,t)$ when $h \to 0$ is given by the optimal consumption over wealth ratio when all wealth is liquid (Merton 1971) and is given by:

$$
\begin{aligned}
q^{Mert} &= \left(m^{-1}\left(e^{mT}-1\right)+\beta^{1/\gamma}e^{mT}\right) \\
m &= \frac{\kappa}{\gamma} - r\left(1-\frac{1}{\gamma}\right) - \frac{1}{2}\left(\frac{\alpha-r}{\sigma}\right)^2\left(\frac{1}{\gamma}-\frac{1}{\gamma^2}\right)
\end{aligned}
$$

Notice that the value of $q(z,t)$ remains finite (although not constant) as the liquid wealth goes to zero, implying that the agent will consume a fraction of its implicit total wealth, even if he can consume only the liquid asset.

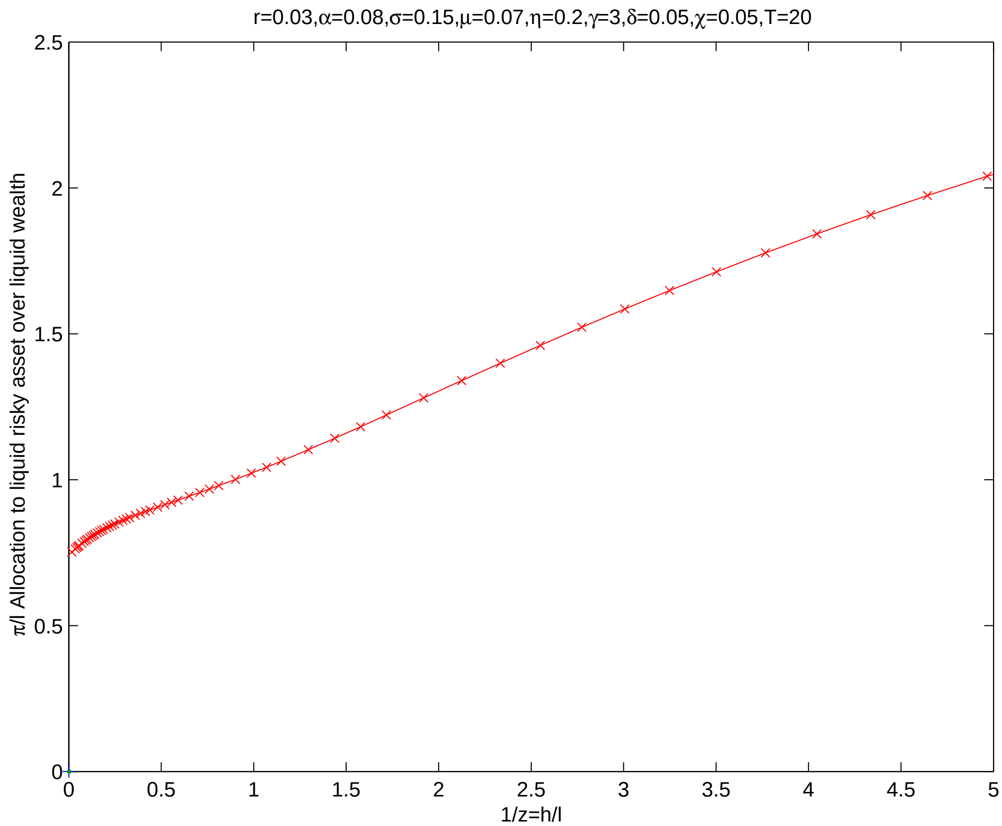

Figure 3: Fractional allocation to the risky liquid asset as a function of $1/z = h/l$.

4.3 Optimal allocation strategy

The optimal allocation is obtained by performing the maximization in (5). Then, expressing the value function in terms of the reduced value function $V(z,t)$, the optimal allocation to the risky liquid asset can be written as:

$$\pi^S = \left[\frac{(\alpha - r)}{\sigma^2} \left(\frac{1}{R(z,t)} \right) - \frac{\eta\rho}{\sigma} \left(\frac{\gamma}{R(z,t)} - 1 \right) \right] l \tag{15}$$

where $R(z,t)$ is defined as:

$$R(z,t) = -\frac{zV_{zz}(z,t)}{V_z(z,t)} \tag{16}$$

Figure 3 shows the fraction of liquid wealth allocated to the liquid risky asset. When $h/l = 1/z \to 0$, $R(z,t)$ converges to the risk aversion parameter γ,

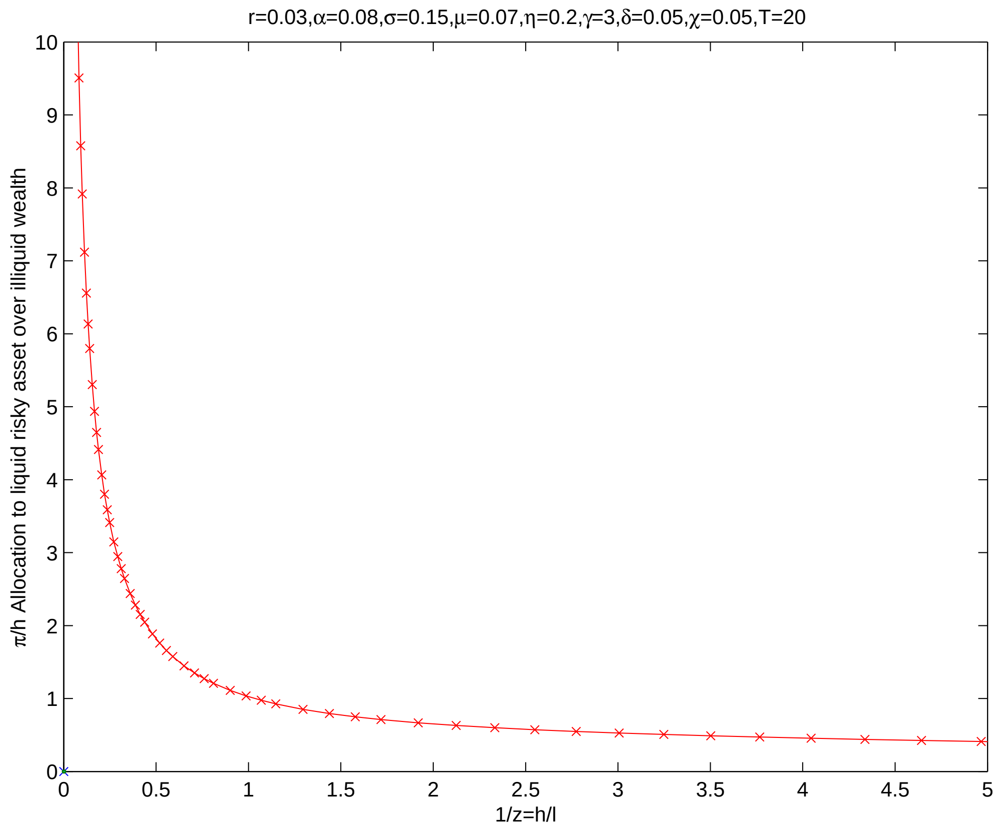

Figure 4: Allocation to the liquid risky asset per unit of illiquid wealth as a function of the $1/z = h/l$.

hence the first term in eq.(15) becomes Merton's (1971) , while the second term, the intertemporal hedging component induced by the presence of the illiquid asset, disappears. When $h/l \to \infty$, $R(z,t) \to 0$ [d] therefore when the agent has a very small amount of liquid wealth, he will invest a large fraction of his liquid holdings in the risky asset in order to diversify the risk induced by the illiquid asset. Figure 4 shows the absolute (dollar) amount invested in the liquid risky asset while keeping constant the nominal value invested in the illiquid asset. Since $\gamma > 1$ and marginal utility is diverging for $h/l \to \infty$, the dollar allocation decreases and converges to 0 when $l \to 0$, while it diverges when $l \to \infty$.

[d] The convergence to this limit has been numerically verified and occurs for $h/l >> 5$.

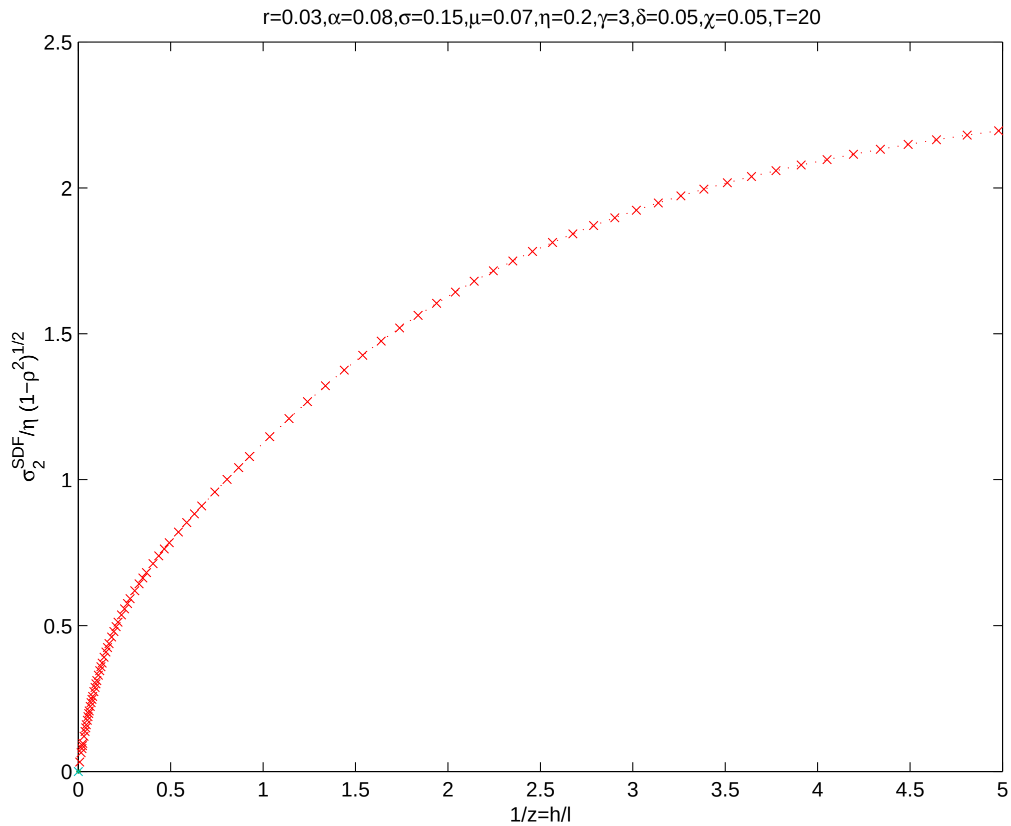

Figure 5: Risk adjustment over the non traded source of risk as a function of $1/z = h/l$

5 The stochastic discount factor

The dual approach that we take allows us to compute the Stochastic Discount Factor (SDF) that the investor uses to optimally value liquid and illiquid assets.

In the continuous time model we are considering the no arbitrage condition implies the existence of a set of Stochastic Discount Factors (SDF), ξ_τ, such that price at time t of a liquid asset that pays a cash flow X_T at time T is given by:

$$X_t = \frac{1}{\xi_t} E_t^{\mathbb{P}} \left[\xi_T X_T \right] \tag{17}$$

where $E_t^{\mathbb{P}}$ denotes the conditional expectation at time t under the measure $\mathbb{P}$. Without loss of generality we can assume that the evolution of the SDF is described by the diffusion:

$$\frac{d\xi_\tau}{\xi_\tau} = \mu^{SDF} d\tau + \sigma_1^{SDF} dW_\tau^1 + \sigma_2^{SDF} dW_\tau^1 \tag{18}$$

Then imposing the condition that eq.(17) hold for the riskless asset and the risky liquid asset we obtain $\mu^{SDF} = -r$ and $\sigma_1^{SDF} = -(\alpha - r)/\sigma$. The presence of an illiquid asset that depends on a source of risk which cannot be diversified trading in the liquid market implies that the SDF will depend also on W_τ^2 while leaving σ_2^{SDF} undetermined. Note that any choice of σ_2^{SDF} will determine a different risk adjusted processes for cash flows whose evolution depend on the risk source W_τ^2 and therefore to different valuations of the illiquid asset.

As discussed in Teplà (2000) and in He and Pearson (1991, Th.7, pg.287) the martingale duality approach to stochastic portfolio optimization provides, under suitable technical assumptions, a feedback formula for the optimal σ_2^{SDF} which the agent uses to value the illiquid asset. In the following proposition we extend their result to our framework:

Proposition 2 *At time t the (indirect) marginal utility of liquid wealth $\lambda = V_l(l, h, t)$ is related to the reduced dual variable y by:*

$$\lambda = yh^\gamma$$

Since at optimality the current marginal utility of consumption equals the indirect marginal utility of liquid wealth, the process for the marginal utility of consumption can be expressed as $\lambda\xi_\tau$, $\tau \geq t$ and the evolution of the SDF is given by equation (18).
The optimal risk adjustment σ_2^{SDF} on the unspanned source of risk W_t^2 is given by:

$$\sigma_2^{SDF}(y,t) = \left(\gamma - \widetilde{R}(y,t)\right)\eta\left(1-\rho^2\right)^{1/2} \tag{19}$$

where $\widetilde{R}(y,t)$ be defined as:

$$\widetilde{R}(y,t) = -\frac{\widetilde{V}_y(y,t)}{y\widetilde{V}_{yy}(y,t)}$$

Hence at optimality the agent will privately value liquid and illiquid cash flows using the SDF:

$$\frac{d\xi_\tau}{\xi_\tau} = -rd\tau - \frac{\alpha - r}{\sigma} dW_\tau^1 - \left(\gamma - \widetilde{R}\left(\lambda\xi_\tau H_\tau^{-\gamma}, \tau\right)\right)\eta\left(1-\rho^2\right)^{1/2} dW_\tau^2$$

where the process for the H_τ is given by eq.(1)

Proof. The proof of this proposition is reported in the Appendix. ■

The above result provides the private valuation formula for the cash flows deriving from the illiquid asset:

$$\widehat{h}_t = \frac{1}{\xi_t} E_t^{\mathbb{P}} \left[\xi_T H_T + \int_t^T \xi_\tau \delta H_\tau d\tau \right]$$

The price given by the private valuation corresponds to the price at which the portfolio including the position in the illiquid asset is an optimal unconstrained allocation for the agent.

Notice that writing the expression for $\widetilde{R}(y,t)$ in terms of the primal state variable z we get

$$R(z,t) = \widetilde{R}(y(z,t),t) = -\frac{zV_{zz}(z,t)}{V_z(z,t)}$$

which allows us to express the risk adjustment σ_2^{SDF} as a function of the liquidity of the agent. In Figure 5 we plot the risk adjustment σ_2^{SDF} per unit of unspanned volatility $\eta(1-\rho^2)^{1/2}$. In fact unspanned volatility $\eta(1-\rho^2)^{1/2}$ can be considered as a measure of market incompleteness, the larger it is the more important the role of non-marketed risks. For a fixed level of unspanned volatility, we can immediately relate σ_2^{SDF} to the risk aversion of the agent. When $h/l \to 0$, $R(z,t) \to \gamma$, thus $\sigma_2^{SDF} \to 0$ and the agent will value his illiquid holdings ignoring the non-traded risk. In the limit when $\sigma_2^{SDF} = 0$ only market risks are valued and the corresponding (marginal) value of the illiquid asset will be:

$$\begin{aligned} \widehat{h}_t^{\sigma_2^{SDF}=0} &= h_t \left[e^{-\nu(T-t)} + \delta\nu^{-1} \left(1 - e^{-\nu(T-t)}\right) \right] \\ \nu &= -\left(\mu - \eta\rho\frac{(\alpha - r)}{\sigma} - \delta - r \right) \end{aligned}$$

This point corresponds to the $h/l = 0$ limit in Figure 1.

When the agent liquid holdings become small and $h/l \to \infty$ then $R(z,t) \to 0$, thus $\sigma_2^{SDF} \to \gamma\eta(1-\rho^2)^{1/2}$. This upper bound for σ_2^{SDF} is reached when the agent only owns the illiquid asset and corresponds to the risk adjustment on the unspanned source of risk required for the agent to optimally hold only the illiquid asset (see Gerber and Shiu 2000).

6 Sensitivity analysis

Figure 6 and Figure 7 analyze the consumption choices of the agent as a function of his liquidity for different times to the final horizon. If no illiquid

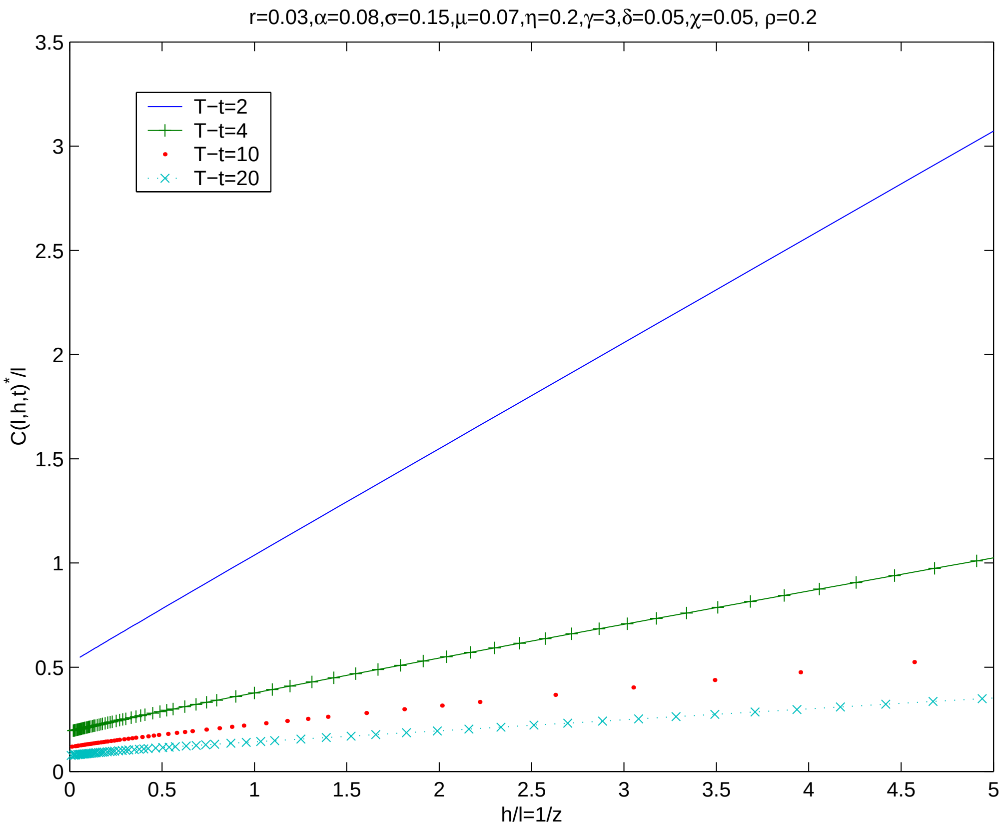

Figure 6: Consumption over liquid wealth ratio as a function of $h/l = 1/z$ for different times to liquidation

asset is held by the agent, $h/l = 0$, and the standard Merton consumption over wealth ratio is recovered in both plots. Figure 6, which considers the consumption per unit of liquid wealth as a function of h/l, shows that an increase in the proportion of the illiquid asset held by the agent will increase his consumption almost linearly. In Figure 7 we consider consumption per unit of total implicit wealth. It shows that consumption over implicit total wealth ratio has little variation and it converges to a constant as $h/l \to \infty$. Note that consumption increases as the final horizon is approached.

If we interpret the illiquid asset as human wealth (the discounted value of future labor income) then our findings can be related to the recent results in the consumption literature (see e.g. Wang (2006) and the references therein). In particular, under the more realistic assumption of constant relative risk aversion preferences for the agent, we can analyze and provide analytical support for both the standard and generalized Friedman (1957) Permanent

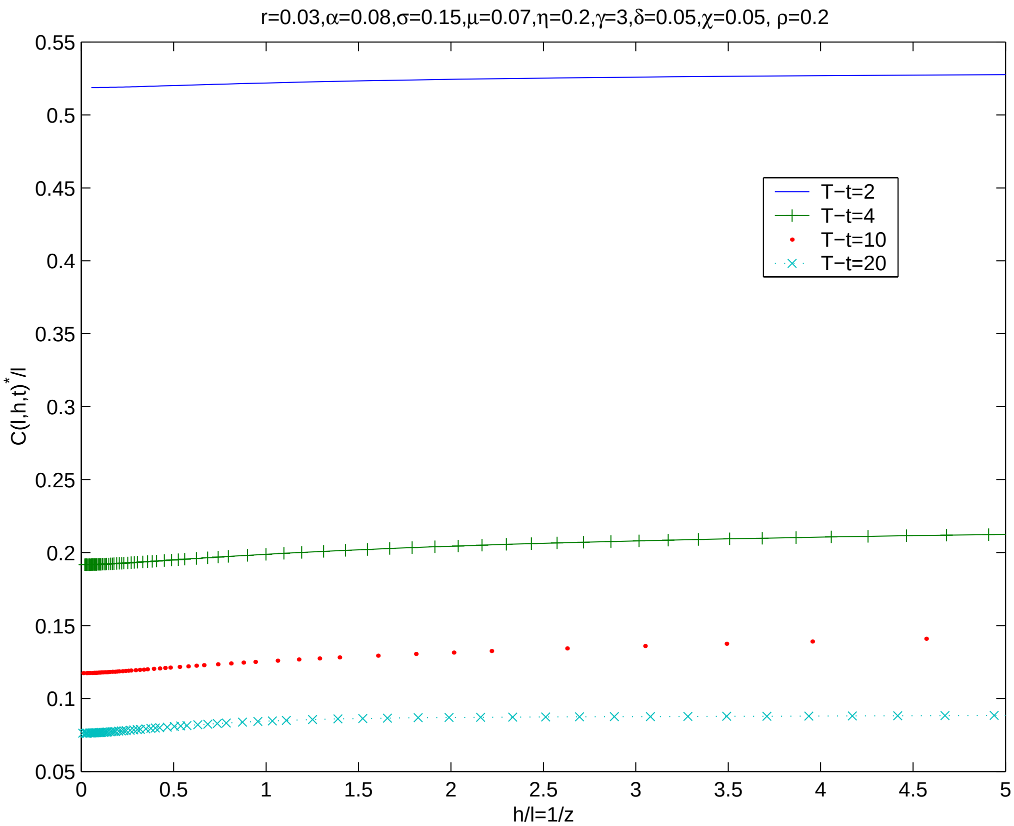

Figure 7: Consumption over implicit total wealth ratio as a function of $h/l = 1/z$ for different times to liquidation

Income Hypotheses (PIH) as defined in Wang (2006). Within this framework, the state variable z determines the fraction of financial wealth (cumulative savings) to human wealth. Assume that the agent values his (implicit) total wealth $w(l, h, t)$ according to formula (13). Then according to PIH, the agent will consume a constant fraction of $w(l, h, t)$. Our model supports this prediction. In fact Figure 7 shows that the ratio of consumption to implicit total wealth has little variation with respect to the liquidity state of the agent as measured by h/l and converges to a constant for h/l sufficiently large.

Moreover, the difference between the elasticity of consumption with respect to the liquid and to the illiquid wealth can be expressed in terms of the function $R(z, t)$:

$$\frac{l}{c^*(l, h, t)} \frac{\partial c^*(l, h, t)}{\partial l} - \frac{h}{c^*(l, h, t)} \frac{\partial c^*(l, h, t)}{\partial h} = \frac{R(z, t)}{\gamma} > 0 \qquad (20)$$

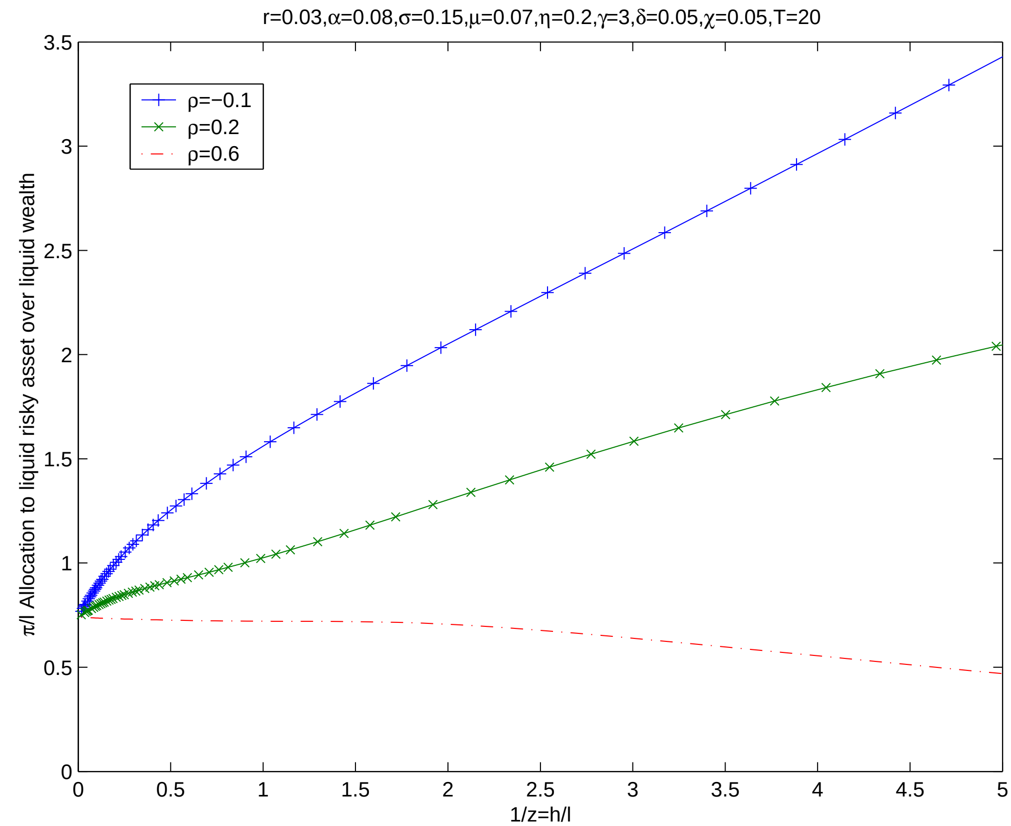

Figure 8: Fractional allocation to the risky liquid asset as a function of $z = l/h$ for different levels of correlation ρ

which shows that the propensity to consume out of financial wealth will be always larger than out of "human wealth". This is the generalized PIH as discussed in Wang (2006). The difference decreases from 1 to 0 as liquidity decreases. When the agent has little or zero savings (financial wealth) he will mostly consume out of his wage, hence out of human wealth.

Figure 8 analyzes the allocation to the risky liquid asset per unit of liquid wealth as a function of liquidity, for different values of the correlation between the liquid and the illiquid asset returns. Note that when we increase h the total wealth of the investor increases also. From the figure we see that the allocation to the liquid risky asset increases for negative correlation and also for moderately positive correlation as well, while for high correlation this allocation decreases. For sufficiently low correlation the diversification effect of having the illiquid asset (even if it cannot be traded) increases the optimal allocation to the liquid risky asset, even to the point of borrowing at the risk free asset to invest in the liquid risky asset. Only when the correlation is sufficiently high and the diversification effect of holding the illiquid risky asset diminishes, does the optimal allocation to the liquid risky asset decreases when h/l increases. This analysis indicates that the professor, with his labor income having low correlation with the risky asset (stock market), should invest more in the risky asset than the stock broker, who has a wage highly correlated with the market.

7 Summary and conclusions

We study the problem of optimal asset allocation in the presence of an illiquid asset. The illiquid asset cannot be traded, but it generates a liquid dividend that can be consumed or invested in liquid assets. This liquid dividend has many interpretations depending on the nature of the illiquid asset. An important application is when the illiquid asset is human wealth and the dividend is labor income. There is a vast literature in economics and finance trying to understand the effect of stochastic labor income on optimal consumption and asset allocation. We obtain closed form solution to this problem in the case of time separable power utility of consumption and terminal wealth.

An important by-product of our analysis is that we derive a valuation procedure for liquid and illiquid assets. In particular, we are able to compute the value that the agent assigns to the illiquid asset, that is, the shadow price of illiquidity. The framework allows, given the preferences of the investor, to value any contingent claim on the illiquid asset or on both the liquid and illiquid asset.

The approach we develop can also be used to solve the optimal asset allocation problem in the presence of borrowing and short selling constraints as discussed in general terms by He and Pages (1993) and Cuoco (1997). In particular, it would be interesting to study the effect that these constraints have on the value that the agent assigns to his illiquid asset.

Perhaps the most challenging extension of our analysis is market equilibrium. If the risky liquid asset is the market portfolio, and the illiquid asset of each agent in the economy is its human wealth, the aggregation problem involves heterogeneous valuations of human wealth holdings of all the agents in the economy. The possibility of asymmetric information effects raises the issue of the impact of moral hazard and adverse selection on such market equilibrium.

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8 Appendix

8.1 Proof of Proposition 1: Series expansion solution of the dual HJB

We will show that after the insertion of eq.(12), the HJB equation (10) becomes equivalent to a system of linear ODE with respect to time.

8.2 The zero order term

Identifying the terms proportional to y^b in the HJB and observing that $\left(y\frac{\partial}{\partial y}\right)y^b = by^b$, we immediately obtain that (10) implies the following ODE for $B_0(t)$:

$$\begin{aligned} \frac{d}{dt}B_0(t) &= -a_0 B_0(t) - \frac{1}{b} \\ B_0(T) &= -\frac{\beta^{1/\gamma}}{b} \end{aligned}$$

where

$$\begin{aligned} a_0 &= \left(K_1 - (K_2 + K_1)\,b + K_3\left(b^2 - b\right) - K_4\frac{b^2}{b^2 - b}\right) \\ &= \left(r + \frac{\kappa - r}{\gamma} - \frac{1}{2}\frac{(1-\gamma)}{\gamma^2}\left(\frac{(\alpha - r)}{\sigma}\right)^2\right) < 0 \end{aligned}$$

this term corresponds to the standard Merton solution which would be exact in absence of the illiquid asset. This equation is a linear non homogenous ODE and can be explicitly solved:

$$B_0(t) = \left\{ \exp\left[a_0 (T-t)\right] \left(\beta^{1/\gamma} + \frac{1}{a_0}\right) - \frac{1}{a_0} \right\} \left(-\frac{1}{b}\right)$$

8.3 First order term

Repeating the procedure for the terms proportional to δy we get an ODE for

$$\begin{aligned} \frac{d}{dt} B_1(t) &= -a_1 B_1(t) + \delta \\ B_1(T) &= +1 \end{aligned}$$

where:

$$\begin{aligned} a_1 &= (2\gamma K_4 - K_2) \\ &= -\left(+(\alpha - r)\frac{\eta\rho}{\sigma} - (\mu - \delta - r)\right) < 0 \end{aligned}$$

and correspondingly the solution will be

$$B_1(t) = \exp\left[a_1 (T-t)\right] \left(1 + \frac{\delta}{a_1}\right) - \frac{\delta}{a_1}$$

8.4 N-th order correction

Now we derive the ODE for the coefficient $B_N(t)$ using a recursive procedure.

First of all notice that:

$$\begin{aligned} y\frac{\partial}{\partial y} y^{b+n/\gamma} &= \left(b + \frac{n}{\gamma}\right) y^{b+n/\gamma} = C_n y^{b+n/\gamma} \\ y^2 \frac{\partial}{\partial y^2} y^{b+n/\gamma} &= \left[\left(b+\frac{n}{\gamma}\right)^2 - \left(b + \frac{n}{\gamma}\right)\right] y^{b+n/\gamma} = D_n y^{b+n/\gamma} \end{aligned}$$

then inserting the trial solution in the ODE system we obtain:

$$\begin{aligned} 0 \; = \; & \left\{ \sum_{n=0}^{\infty} \left[\partial B_n(t)/\partial t + \left(K_1 - (K_2 + K_1) C_n + K_3 D_n\right) B_n(t)\right] y^{b+n/\gamma} + F(y) \right\} \\ & \left\{ \sum_{n=0}^{\infty} D_n B_n(t)\, y^{b+n/\gamma} \right\} - K_4 \left\{ \sum_{n=0}^{\infty} C_n B_n(t)\, y^{b+n/\gamma} \right\}^2 \end{aligned}$$

and collecting the terms of the same order we get that the coefficient of the $m-th$ order term $(y)^{2b+m/\gamma}$ $m \geq 2$ will be:

$$\sum_{n=0}^{m} D_{m-n}B_{m-n}(t)\left[\partial B_n(t)/\partial t + (K_1 - (K_2+K_1)C_n + K_3D_n)B_n(t)\right] +$$
$$-K_4\sum_{n=0}^{m} C_{m-n}B_{m-n}(t)\,C_nB_n(t)$$

In order to satisfy the HJB equation each of these coefficients has to be equal to 0. Elementary algebra shows that if we consider the equation for the coefficients up to order N, $\{B_n(t)\}_{n=0,\ldots N}$, then the equation to be satisfied by $B_N(t)$ is given by:

$$\begin{aligned} 0 = &\left[\partial B_N(t)/\partial t + \left[K_1 - (K_2+K_1)C_N + K_3D_N\right]B_N(t)\right] \\ &-K_4\sum_{m,l=0}^{N} M_{N+1,m+1}G_{m+1,l+1}v_{l+1} \\ B_0(T) = &-1/b, \quad B_1(T)=1, \quad B_n(T)=0 \quad 2\leq n\leq N \end{aligned}$$

where:

$$\begin{aligned} M &= (Tp)^{-1} \\ Tp_{m+1,l+1} &= D_{m-l}B_{m-l}(t)\,,\ m\geq l \quad Tp_{m+1,l+1}=0, \quad m<l, \quad \dim T = N+1 \\ G_{m+1,l+1} &= C_{m-l}B_{m-l}(t)\,,\ m\geq l \quad G_{m+1,l+1}=0, \quad m<l, \quad \dim G = N+1 \\ v_{l+1} &= C_lB_l, \quad l=0,..,N \end{aligned}$$

The matrices we introduced are called "Toeplitz matrices". These lower triangular matrices are characterized by the property that its elements are constant along diagonals e.g.

$$Tp = \begin{bmatrix} D_0B_0(t) & 0 & 0 \\ D_1B_1(t) & D_0B_0(t) & 0 \\ D_2B_2(t) & D_1B_1(t) & D_0B_0(t) \end{bmatrix}$$

The inverse of a Toeplitz matrix will also be a lower triangular Toeplitz matrix.

This system of equations is highly non linear due to the last term. A crucial property coming from the triangularity of the matrices is the following: the ODE equation for the highest order coefficient $B_N(t)$, given the coefficients $\{B_n(t)\}_{n=0,\ldots,N-1}$ is a linear equation; in fact at level N the equation will be:

$$
\begin{aligned}
0 &= \left[\partial B_N(t)/\partial t + \left(K_1 - \left(K_2+K_1\right)C_N + K_3 D_N\right) B_N(t)\right] \\
&\quad - K_4 \sum_{m,l=0}^{N} M_{N+1,m+1} G_{m+1,l+1} v_{l+1}
\end{aligned}
$$

and extracting from the last summation the terms which contain B_N we get:

$$
\begin{aligned}
0 &= \partial B_N(t)/\partial t - K_4 f^{(N)}(t) + \\
&\quad \left[K_1 - \left(K_2 + K_1 - \frac{2K_4}{b-1}\right) C_N + \left(K_3 - \frac{K_4}{(b-1)^2}\right) D_N\right] B_N(t) \\
f^{(N)}(t) &= \sum_{m=0}^{N}\sum_{l=0}^{N} M^*_{N+1,m+1} G^*_{m+1,l+1} v^*_{l+1} \qquad (21) \\
M^* &= (Tp^*)^{-1}, \\
(Tp)^*_{N+1,1} &= 0,\ \ (Tp)^*_{m+1,n+1} = (Tp)_{m+1,n+1} \qquad (m,n) \neq (N,0) \\
G^*_{N+1,1} &= 0,\ \ G^*_{m+1,n+1} = G_{m+1,n+1} \qquad (m,n) \neq (N,0) \\
v^*_{N+1} &= 0,\ \ v^*_{l+1} = v_{l+1} \qquad l \neq N
\end{aligned}
$$

Then the linear ODE to be solved is:

$$
\begin{aligned}
\partial B_N(t)/\partial t &= -a_N B_N(t) + K_4 f^{(N)}(t) \\
B_N(T) &= 0
\end{aligned}
$$

where

$$
a_N = \left[K_1 - \left(K_2 + K_1 - \frac{2K_4}{b-1}\right) C_N + \left(K_3 - \frac{K_4}{(b-1)^2}\right) D_N\right] \qquad (22)
$$

and

$$
f^{(N)}(t) = \sum_{m=0}^{N}\sum_{l=0}^{N} M^*_{N+1,m+1} G^*_{m+1,l+1} v^*_{l+1} \qquad (23)
$$

The general solution to this equation will be:

$$
B_N(t) = -K_4 \int_t^T e^{a_N(s-t)} f^{(N)}(s)\, ds
$$

8.5 Computation of partial derivatives

$$\begin{aligned}
V_t(l,h,t) &= -\kappa e^{-\kappa t} h^{1-\gamma} V(z,t) + e^{-\kappa t} h^{1-\gamma} V_t(z,t) \qquad (24)\\
V(l,h,t) &= e^{-\kappa t} h^{1-\gamma} V(z,t)\\
V_l(l,h,t) &= e^{-\kappa t} h^{-\gamma} V_z(z,t)\\
V_{ll}(l,h,t) &= e^{-\kappa t} h^{-\gamma-1} V_{zz}(z,t)\\
V_h(l,h,t) &= e^{-\kappa t} h^{-\gamma} \left[(1-\gamma) V(z,t) - z V_z(z,t)\right]\\
V_{hh}(l,h,t) &= e^{-\kappa t} h^{-\gamma-1} \left[-\gamma(1-\gamma) V(z,t) + 2\gamma z V_z(z,t) + z^2 V_{zz}(z,t)\right]\\
V_{lh}(l,h,t) &= -e^{-\kappa t} h^{-\gamma-1} \left[\gamma V_z(z,t) - z V_{zz}(z,t)\right]
\end{aligned}$$

8.6 Proof of equation (14)

If we define q as:

$$q := \left[\frac{(1-\gamma) V(l,h,t)}{h^{1-\gamma}(z+p(z,t))^{1-\gamma}}\right]^{-1/\gamma}$$

homogeneity of V implies that q is a function of z and t only, hence we can represent $V(l,h,t)$ and $V(z,t)$ as:

$$\begin{aligned}
V(l,h,t) &= q(z,t)^{-\gamma} \frac{w(l,h,t)^{1-\gamma}}{1-\gamma}\\
V(z,t) &= e^{\kappa t} q(z,t)^{-\gamma} \frac{(z+p(z))^{1-\gamma}}{1-\gamma}
\end{aligned}$$

By eq.(5) optimal consumption is given by:

$$c^*(l,h,t)^{-\gamma} = V_l(l,h,t) = e^{-\kappa t} h^{-\gamma} V_z(z,t)$$

and by the definition of $p(z,t)$:

$$V_z(z,t) = (1-\gamma)\frac{V(z,t)}{z+p(z)} = e^{\kappa t} q(z,t)^{-\gamma}(z+p(z))^{-\gamma}$$

hence the consumption over wealth ratio will be defined by:

$$\begin{aligned}
\frac{c^*(l,h,t)}{w(l,h,t)} &= \frac{\left(e^{-\kappa t} V_z(z,t)\right)^{-1/\gamma}}{z+p(z)}\\
&= \frac{\left(q(z,t)^{-\gamma}(z+p(z))^{-\gamma}\right)^{-1/\gamma}}{z+p(z)}\\
&= q(z,t)
\end{aligned}$$

8.7 Proof of equation (15)

The maximization in eqs.(5) yields the risky allocation formula:

$$-\frac{V_l(l,h,t)}{V_{ll}(l,h,t)}\frac{\alpha-r}{\sigma^2}-\frac{hV_{lh}(l,h,t)}{V_{ll}(l,h,t)}\frac{\eta\rho}{\sigma}$$

and applying eqs.(24) we obtain:

$$\left[\frac{\alpha-r-\gamma\eta\rho\sigma}{\sigma^2}\left(-\frac{V_z(z,t)}{V_{zz}(z,t)}\right)-\frac{\eta\rho}{\sigma}z\right]h$$

and using the definition of $R(z,t)$ we conclude.

8.8 Proof of Proposition 2

The proof is an immediate consequence of He and Pearson (1991) Th.7 and similar to Proposition 1 inTeplà (2000). Th.7 states the following: let $l(\lambda,h,t)$ be the liquid wealth considered as a function of h, t and $\lambda=V_l(l,h,t)$, the conjugate variable of l, then following feedback representation for the the volatility of the SDF, i.e. optimal risk adjustment on W_t^2, v^*, holds:

$$v^*(\lambda,h,t)=\frac{hl_h(\lambda,h,t)}{\lambda l_\lambda(\lambda,h,t)}\eta\left(1-\rho^2\right)^{1/2}$$

Since we apply duality to the reduced value function $V(z,t)$, and to the rescaled liquid wealth $z=l/h$, it is straightforward to prove that homogeneity implies the relation: $\lambda=yh^{-\gamma}$. In terms of reduced variable y and h, the feedback formula for v^* will be:

$$v^*(\lambda(y,h),h,t)=\left[\gamma-\frac{l(\lambda(y,h),h,t)}{yl_y(\lambda(y,h),h,t)}\right]\eta\left(1-\rho^2\right)^{1/2}$$

Since $l(\lambda(y,h),h,t)=hz(y,t)$ where $z(y,t)=-\widetilde{V}_y(y,t)$ then we can conclude that:

$$\sigma_2^{SDF}(y,t)=v^*(y,1,t)=\left(\gamma-\frac{\widetilde{V}_y(y,t)}{y\widetilde{V}_{yy}(y,t)}\right)\eta\left(1-\rho^2\right)^{1/2}$$

and (19) holds.

8.9 Proof of equation (20)

Optimal consumption is given by (5) hence: $c^*(l,h,t) = V_l(l,h,t)^{-1/\gamma}$. In terms of the state variables h, z and t we have:

$$
\begin{aligned}
l(h,z,t) &= hz \\
c(l(z,h,t),h,t)^* &= hV_z(z,t)^{-1/\gamma}
\end{aligned}
$$

Then the elasticity of consumption w.r.t liquid wealth can be written as:

$$
\begin{aligned}
\frac{l(z,h,t)}{c^*(l(z,h,t),h,t)}\frac{\partial c^*((z,h,t),h,t)}{\partial l(z,h,t)} &= \frac{zh}{hV_z(z,t)^{-1/\gamma}}\frac{\partial\left(hV_z(z,t)^{-1/\gamma}\right)}{\partial z}\frac{\partial z}{\partial l(z,h,t)} \\
&+\frac{l}{hV_z(z,t)^{-1/\gamma}}\frac{\partial\left(hV_z(z,t)^{-1/\gamma}\right)}{\partial h}\frac{\partial h}{\partial l(z,h,t)}
\end{aligned}
$$

which equals

$$
\frac{l}{c^*(l,h,t)}\frac{\partial c^*(l,h,t)}{\partial l(l,h,t)} = -\frac{1}{\gamma}\frac{zV_{zz}(z,t)}{V_z(z,t)} + \frac{h}{c^*(l,h,t)}\frac{\partial c^*(l,h,t)}{\partial h}
$$

hence by eq.(16) we can conclude:

$$
\frac{l}{c^*(l,h,t)}\frac{\partial c^*(l,h,t)}{\partial l} - \frac{h}{c^*(l,h,t)}\frac{\partial c^*(l,h,t)}{\partial h} = \frac{R(z)}{\gamma}
$$

Notes

1. Cuoco (1997) provides necessary conditions for the existence of an optimal solution to the present stochastic optimization problem, provided that the set of admissible strategies is restricted by the following technical conditions on the processes:

$$(c, W) \in \mathcal{C} = \left\{ \begin{array}{c} c_t \;\; \geq 0 \; a.s. \; \mathcal{F}_t\text{-adapted}, \; \int_0^T |c_t| \, dt < +\infty, \; t \leq T \\ W_T^{\pi,c} = H_T + L_T^{\pi,c} \geq 0 \; a.s. \; W_T^{\pi,c} \in \mathcal{L}^2 \left(\mathcal{F}_T \right) \end{array} \right\}$$

while the set of admissible strategies $\left(\pi_t^B, \pi_t^S\right)$ has to obey:

$$\int_0^T \left|\pi^B\right| r dt + \int_0^T \left|\pi^S\right| \alpha dt + \int_0^T \left|\pi^S\right|^2 \sigma^2 dt < \infty.$$

2. Notice that when $\gamma > 1$ these requirements are sufficient to guarantee that, starting with strictly positive liquid wealth, it will never be optimal to reach negative liquid wealth. This is suggested by the following informal argument: suppose on the contrary that a negative position in liquid wealth is possible, then there would be a small but non vanishing probability for the final total wealth to be negative but this is prevented by the fact that marginal utility is diverging at zero total wealth.

3. A large strand of mathematical literature has concentrated on establishing existence and regularity of the solution to the stochastic optimization problem of the consumption-investment problem, see e.g. Cuoco (1999), Karatzas and Zikovic (2003). We focus on the computational approach to solve the PDE equation and do not provide verification theorems, proofs of existence and regularity of the HJB solutions. For this reason we consider the regularity conditions of the value function as working hypotheses; in particular we conjecture that the primal (dual) value function is concave and increasing (convex and decreasing) with continuous first and second order derivatives which never vanish in the domains $x > 0$, $y > 0$. In the numerical implementation these hypotheses have been checked on the truncated series solution, which has been considered valid only in the effective domains where these conditions apply.

4. This expression for the function $\widetilde{V}(y,t)$ has been derived using a scaling argument for the infinite horizon situation $T = \infty$: since h affects the optimization problem only through the dividend, then it is easy to observe that

the transformation $\delta \to \frac{\delta}{\alpha}$, $h \to \alpha h$, for fixed price appreciation of the illiquid asset, leaves the optimization problem unaffected (the same argument can be easily extended to the finite horizon case). This implies that:

$$\widetilde{V}(y,t) = y^{1-1/\gamma}\phi\left(\delta y^{1/\gamma}\right)$$

where $\phi(x)$ is usually called "scaling function". It is typical in the analysis of critical phenomena and renormalization, see e.g. Stanley (1999), to assume that the scaling function is an analytical function of its argument. Further mathematical research, is needed in order to provide a rigorous analysis of the convergence properties of the series

5. The (transversality) conditions on the coefficients $a_0 < 0$ and $a_1 < 0$ are required in order to exclude bubbles and ensure the existence of a stable limit $T \to \infty$

6. Pointwise convergence of the function series expansion (12) implies that for a given y^* the numerical series is convergent. From the numerical point of view this implies that the summation of the truncated series becomes largely independent from the truncation order for sufficiently large number of terms. On the other hand our main hypothesis that $\phi(x)$ is analytic, implies that we can define the analytic continuation of the function series expansion (12) outside of its domain of convergence. In order to obtain a better evaluation of ϕ we consider the analytic continuation of the series (12) as defined in terms of its Borel Transform, see e.g. Whittaker and Watson (1990), then we use it to analyze the behavior of ϕ. We consider the solution acceptable in the domain where our working hypotheses on differentials of the value functions are verified. Inspection of numerical results show that the approximation to the analytical function fails above a given $y_{\max}$ (below the corresponding $z_{\min}$) with a sharp transition. These approximation problems arise in correspondence to a minimum level $z_{\min}$ which is usually of order 0.1.